SURFACES WHOSE GEODESICS MAY BE REPRESENTED IN THE PLANE BY PARABOLAS*

BY

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The general problem of geodesic representation, whose preliminary discussion was given by the author in the Transactions for 1903,† is essentially the the same as what may be termed the *inverse problem of geodesics,* proposed by Lie‡ as follows: Given the equation of the geodesics in the form $F(u, v, \lambda, \mu) = 0$, determine the corresponding surfaces.

For the existence of such a surface, it is necessary that the differential equation of given system should be of the form

$$(1) \qquad y'' = Ay'^3 + By'^2 + Cy' + D,$$

where the coefficients are functions of x, y. We shall refer to any equation of this form as one of *cubic type.* Such equations are of interest independently of their connection with geodesics: the type is invariant with respect to arbitrary point transformation.

In order that (1) shall represent a system of possible geodesics, it is further necessary that the functions A, B, C, D satisfy certain relations expressed by the consistency of a system of partial differential equations.§ For our purpose, it is not necessary to have these relations explicitly. We note merely that the equations of *geodesic type* constitute only a subclass of the general cubic class (1) — a subclass which is itself invariant under the group of all point transformations.

A more special subclass is formed by those equations of the second order which are equivalent, under point transformation, to $y'' = 0$. Such an equation may be characterized by the fact that its complete integral can be put into the form

$$f_1(x, y) + \text{const.} f_2(x, y) + \text{const.} f_3(x, y) = 0,$$

of the first degree in the two parameters. This subclass includes then all linear systems, in the general sense of the term. It is a part of the geodesic class, corresponding, by BELTRAMI's theorem, to the surfaces of constant curvature. * The necessary and sufficient conditions on the·coefficients of (1), for such a system, are †

$$(BD - D_y)_y - (AD + \tfrac{1}{3}B_x - \tfrac{2}{3}C_y)_x + C(AD + \tfrac{1}{3}B_x - \tfrac{2}{3}C_y)$$
$$- D(AC + A_x) = 0,$$

$$-(AC + A_x)_x + (AD + \tfrac{2}{3}B_x - \tfrac{1}{3}C_y)_y + B(AD + \tfrac{2}{3}B_x - \tfrac{1}{3}C_y)$$
$$- A(BD - D_y) = 0.$$

In the present paper we consider problems of this kind: Given a system of plane curves involving three or more parameters, determine the two-parameter systems capable of representing the geodesics of a surface, and investigate the corresponding surfaces. A more general problem is the determination of all the integrals of form (1) satisfying a given differential equation of the third or higher order.‡ The only case that has received treatment is the case of circles; here the only systems of geodesic type are linear, so that the corresponding surfaces are merely those of constant curvature. §

The main part of the paper, §§ 1–7, is devoted to what seems to be the simplest case of our problem, namely, the three-parameter system of vertical parabolas

(2) $$y = \lambda x^2 + \mu x + \nu,$$

or the corresponding differential equation

(3) $$y''' = 0.$$

We shall find that this leads to interesting and important results; in particular, it is the first case which leads to surfaces other than those of constant curvature.

In § 1, we find all the doubly infinite systems of parabolas whose differential equation is of the cubic form (1). In addition to linear systems, certain quadratic systems satisfy the conditions of the problem (§ 2). The systems are further classified and characterized geometrically (§ 3, § 4). In § 5 the existence

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* Cf. Transactions, vol. 4 (1903), p. 149.

† R. LIOUVILLE, Journal de l'Ecole Polytechnique, vol. 57 (1887), p. 219. For an application of these equations cf. E. KASNER, *A characteristic property of isothermal systems*, Mathematische Annalen, vol. 59 (1904), p. 352.

‡ In general no solution exists, since the functions A, B, C, D must satisfy an infinite number of conditions. An example where the inconsistency of the conditions may be verified very simply is $y''' = y'^6$. The same holds for equations of higher order.

§ Cf. BUSSE, Berliner Sitzungsberichte, 1896. It may be shown, by an investigation similar to that in § 1 of the present paper, that the only systems of circles of the cubic type (1) are the linear systems.

of corresponding surfaces is verified, namely, those of constant curvature, and those which are geodesically representable on surfaces with the linear element

$$ds^2 = v(du^2 + dv^2).$$

In § 7 the group of the geodesics is determined in order to obtain the most general representation in which geodesics are pictured by parabolas. In § 6 the problem is specialized by adding the requirement that the representation shall be conformal; the surfaces are then either developables or applicable on a certain surface of revolution. Finally, in § 8, certain more general problems in conformal representation are treated; in this section the discussion is merely outlined since no new classes of surfaces are brought to light.

The surfaces of variable curvature which satisfy the conditions of our problem present themselves, in whole or in part, in many important investigations. We mention here the following, detailed references being given later: (1) Systems of geodesics admitting infinitesimal transformations (LIE, KOENIGS). (2) Systems of geodesic circles admitting contact transformations (LIE). (3) Surfaces for which the partial differential equation $\Delta\theta = 1$ admits integrals of the first and second degree in the first derivatives (DARBOUX). (4) Surfaces whose element is reducible to the LIOUVILLE form in an infinite number of ways (RAFFY, KOENIGS).

§ 1. *Differential equations of the problem.*

We first obtain the conditions that an equation of cubic type (1) shall represent a double infinity of parabolas (2). For this purpose we differentiate (1) and find that

(4) $$y''' = \alpha_1 y'^5 + \alpha_2 y'^4 + \alpha_3 y'^3 + \alpha_4 y'^2 + \alpha_5 y' + \alpha_6,$$

where

(5)
$$\alpha_1 = 3A^2, \qquad \alpha_4 = 3AD + 3BC + B_x + C_y,$$
$$\alpha_2 = 5AB + A_y, \qquad \alpha_5 = 2BD + C^2 + C_x + D_y,$$
$$\alpha_3 = 4AC + 2B^2 + A_x + B_y, \qquad \alpha_6 = CD + D_x.$$

Since for our parabolas equation (3) must be satisfied, we have then the conditions

$$\alpha_1 = 0, \ \alpha_2 = 0, \ \alpha_3 = 0, \ \alpha_4 = 0, \ \alpha_5 = 0, \ \alpha_6 = 0.$$

The first of these shows that

(6) $$A = 0;$$

the second is then satisfied identically; and the others give the following four equations for the determination of B, C, D:

$$B_y + 2B^2 = 0,$$

(7)
$$B_x + C_y + 3BC = 0,$$

$$C_x + D_y + C^2 + 2BD = 0,$$

$$D_x + CD = 0.$$

It will be convenient to divide the discussion of this system with reference to the vanishing or non-vanishing of B.

Discussion for $B = 0$.

Putting $B = 0$ into (7_2), we find that C is a function of x alone, say the function X. The remaining equations of the system then give

(8) $$D_y + X' + X^2 = 0, \qquad D_x + XD = 0.$$

Eliminating D, we obtain

$$X'' + 3XX' + X^3 = 0,$$

whose solution is

$$C = X = \frac{2(ax + b)}{ax^2 + 2bx + c}.$$

The value of D is now found, from (7_4), to be

$$D = \frac{-2ay + 2d}{ax^2 + 2bx + c}.$$

These values of C and D, together with $A = 0$, $B = 0$, satisfy all our conditions. The corresponding equation (1) is

(9) $$(ax^2 + 2bx + c)y'' = 2(ax + b)y' - 2(ay - d).$$

Discussion for $B \neq 0$.

The integration of (7_1) gives

(10) $$B = (2y + X)^{-1},$$

where X is an arbitrary function of x. We now simplify the remaining equations of the set (7) by introducing new variables u, v as follows:

(11) $$u = x, \qquad v = (2y + X)^{-\frac{1}{2}}.$$

The result of the transformation, noting that $B = v^2$, is found to be

$$vC_v - 3C + U'v^2 = 0,$$

(12) $$-v^3 D_v + 2v^2 D + C^2 + C_u - \tfrac{1}{2} U'v^3 C_v = 0,$$

$$D_u - \tfrac{1}{2} U'v^3 D_v + CD = 0,$$

where U denotes the result of substituting u for x in the function X.

The solution of the first of these equations is

(13) $$C + U'v^2 + U_1 v^3,$$

where U_1 is an arbitrary function of u. The other equations may now be written

(14) $$D_u = -U_1 v^3 D + \tfrac{1}{2} U' v^2 R,$$
$$D_v = 2v^{-1} D + v^{-1} R,$$

where R is defined by

(14′) $$R = U'' + U_1' v + \tfrac{1}{2} U' U_1 v^3 + U_1^2 v^4.$$

Equating the two values of D_{uv} found from (14), we have

(15) $$3 U_1 D + S = 0,$$
where
(15′) $$S = U_1 R - \tfrac{1}{2} U' R_v + v^{-3} R_u.$$

We now proceed to show that if U_1 does not vanish, the equations (14) are inconsistent. For, under the assumption that $U_1 \neq 0$, (15) may be written

$$D = -\frac{S}{3 U_1}.$$

Substituting this in (14$_2$), we find

$$S_v - 2v^{-1} S + 3 U_1 v^{-1} R = 0.$$

This must be satisfied identically if equations (14) are to be consistent. Substituting the values of R and S given in (14′) and (15′), the coefficient of every power of v must vanish. For our purpose it is sufficient to select the coefficient of v^3, which is found to be $5 U_1^3$. This, however, cannot vanish, in virtue of the assumption made at the outset.

We may therefore assume that

(16) $$U_1 = 0,$$

Then equation (15) gives $S = 0$. From (14′), R reduces to U''; and thereby, from (15′), S reduces to $v^{-3} U'''$. Equating the two values of S thus obtained, we have $U''' = 0$, or

(17) $$U = au^2 + 2bu + c,$$

where a, b, c are arbitrary constants.

Substituting the values of U and U_1 in (14), we find

$$D_u = 2av^2(au + b), \qquad D_v = 2b^{-1}(D + a).$$

From the derivation these equations must be consistent; their common solution, containing a new arbitrary constant d', is in fact

(18) $$D = av^2(au^2 + 2bu + d') - a.$$

The value of C given in (13) reduces, in virtue of (16) and (17), to

(19) $$C = 2v^2(au + b).$$

This completes the discussion of the set (7) for $B \neq 0$. The solution found, given in (10), (19), (18), may be reduced by means of the substitution formulas (11) and the value

$$X = ax^2 + 2bx + c,$$

to the form

(19)
$$B = \frac{1}{2y + ax^2 + 2bx + c},$$

$$C = \frac{2(ax + b)}{2y + ax^2 + 2bx + c},$$

$$D = \frac{-2(ay - d)}{2y + ax^2 + 2bx + c},$$

where d takes the place of $a(d' - c)$.

The differential equation (1), thus found, is

(20) $$(2y + ax^2 + 2bx + c)y'' = y'^2 + 2(ax + b)y' - 2(ay - d).$$

We may state the result obtained at this stage as follows:

The only equations of cubic type (1) *which represent a doubly infinite system of parabolas* (2) *are those of the forms* (9) *and* (20). *Both these forms are included in the complete solution*

(21) $$(2ty + ax^2 + 2bx + c)y'' = ty'^2 + 2(ax + b)y' - 2(ay - d)$$

containing four constants $a : b : c : d : t$.

§ 2. *General character of the systems of parabolas.*

There is no difficulty in integrating (21); for it is known, from the derivation, that the integral is of the form (2). We find that *our systems of parabolas are represented by*

$$y = \lambda x^2 + \mu x + \nu,$$

where λ, μ, ν *are connected by a relation of the form*

(21′) $$t(\mu^2 - 4\lambda\nu) - 2c\lambda + 2b\mu - 2a\nu + 2d = 0.$$

The character of the system, we shall see, depends essentially upon the discriminant of this quadratic in λ, μ, ν, namely,

(22) $$\delta = 4t^2(b^2 - ac - 2td).$$

We show first that, if $\delta = 0$, the equation (21) is equivalent, under the group of point transformations, to $y'' = 0$. This requires the discussion of the cases $t = 0$ and $b^2 - ac - 2td = 0$.

If $t = 0$, that is, if equation (21) is of the form (9), the corresponding relation (21') reduces to the form

(9')
$$- 2c\lambda + 2b\mu - 2a\nu + 2d = 0.$$

This system of parabolas being linear, is evidently reducible to $y'' = 0$.

If t is not 0, we may assume it to be unity, so that our equation is of form (20). The vanishing of δ requires $b^2 - ac - 2d = 0$. The transformation

(23)
$$x_1 = x, \qquad 2y_1 = 2y + ax^2 + 2bx + c$$

then converts (20) into

(24)
$$2y_1 y_1'' = y_1'^2,$$

For this normal form the relation (21') becomes

$$\mu^2 - 4\lambda\nu = 0.$$

We may therefore put

$$\lambda = \alpha^2, \qquad \mu = 2\alpha\beta, \qquad \nu = \beta^2,$$

so that the solution of (24) may be written

(24')
$$y = (\alpha x + \beta)^2.$$

This represents the vertical parabolas touching the axis of x. In the theory of algebraic curves this would be termed a quadratic system since the parameters are involved to the second degree. But in the general (infinitesimal) theory [*] of systems of curves the system (24') is linear since it may be written

$$\alpha x + \beta - \sqrt{y} = 0,$$

which involves the parameters linearly. Hence the equation (24) is equivalent to $y'' = 0$. This completes the proof of the result stated above.

We pass now to the case $\delta \neq 0$. We take, as before, $t = 1$, and put

(25)
$$\kappa^2 = ac - b^2 + d;$$

so that κ, by assumption, does not vanish. The transformation [†]

(26)
$$x_1 = x, \qquad 2\kappa y_1 = 2y + ax^2 + 2bx + c$$

converts (20) into

(27)
$$2y_1 y_1'' = y_1'^2 + 1.$$

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[*] See Bulletin of the American Mathematical Society, vol. 12 (1905), p. 307.

[†] If reality considerations are taken into account, it is necessary to separate the cases $\delta < 0$, and $\delta > 0$. In the latter case, κ is imaginary; but another transformation gives the normal form $2yy'' = y'^2 - 1$. This represents the vertical parabolas whose foci are on the axis of x.

The solution for this normal form is

$$4\beta(y - \beta) = (x - \alpha)^2,$$

which represents the parabolas having the axis of x for directrix.

This is an essentially distinct case; the equation (27) is not equivalent to $y'' = 0$, since it does not satisfy the conditions of LIOUVILLE given in the introduction. We have then this result:

With respect to the group of all point transformation, the systems of parabolas possessing a cubic differential equation (1), *that is, the systems* (21), *divide into two distinct types. In the first type, the discriminant δ of the relation* (21') *vanishes, and the system is linear, i. e., equivalent to $y'' = 0$. In the second type, δ does not vanish, the system is essentially quadratic, and equivalent to* (27).

§ 3. *Detailed classifications.*

The preceding paragraph contains the classification of our systems (21) with respect to the group of all point transformations. We can obtain a closer geometric survey of the systems by means of classifications based on certain finite (continuous) groups which suggest themselves in connection with our problem.

The totality of vertical parabolas (2) evidently admits the projective transformations of the form

$$G_5: \qquad x_1 = e_1 x + e_2, \qquad y_1 = e_3 y + e_4 x + e_5,$$

which constitute a five-parameter group. It may be shown that it admits a more extensive group

$$G_6: \qquad x_1 = e_1 x + e_2, \qquad y_1 = e_0 x^2 + e_3 y + e_4 x + e_5.$$

This group G_6 contains G_5 as a subgroup, the latter arising by putting $e_0 = 0$.

Since any point transformation converts an equation of cubic form (1) into an equation of the same form, and since the particular transformations G_6 also convert vertical parabolas into vertical parabolas, it follows that the transformations G_6 convert any solution of our problem, that is, any system (21), into a solution or system (21). The effect of a transformation G_6 upon the equation (21) containing the constants $t : a : b : c : d$, is to convert it into an equation of the same form with new constants $t_1 : a_1 : b_1 : c_1 : d_1$.

The induced transformation of the constants is found to be

$$G_6': \quad \begin{aligned} t_1 &= e_3^2 t, \\ a_1 &= 2e_3 e_0 t + e_3 e_1 a, \\ b_1 &= e_3 e_4 t + e_3 e_1 e_2 a + e_3 e_1 b, \\ c_1 &= 2e_3 e_5 t + e_3 e_2^2 a + 2e_3 e_2 b + e_3 c, \\ d_1 &= \frac{e_4^2 - 4e_0 e_5}{2} t + (e_1 e_2 e_4 - e_0 e_2^2 - e_1^2 e_5) a + (e_1 e_4 - 2e_0 e_2) b - e_0 c + e_1^2 d. \end{aligned}$$

The induced group G_5', corresponding to the projective transformations G_5, is obtained by simply putting $e_0 = 0$ in the above. The modulus of the linear transformation is $e_1^5 e_3^3$; this cannot vanish, since otherwise the original transformation G_6 would have a vanishing jacobian and thus degenerate. We may therefore assume that neither e_1 nor e_3 vanishes.

We are now in position to discuss the equivalence of our systems of parabolas, or the corresponding differential equations (21), with respect to the group G_6. From the importance of the quantity δ in the general discussion of § 2, one is led to expect its invariant character in the present connection. In fact, the factors of δ give rise to two (relative) invariants t and $b^2 - ac - 2td$; for

$$t_1 = e_3^2 t,$$

$$b_1^2 - a_1 c_1 - 2t_1 d_1 = e_1^2 e_3^2 (b^2 - ac - 2td).$$

This shows that the systems for which $t = 0$ must be distinguished from those for which $t \neq 0$; and, similarly, the cases $b^2 - ac - 2td = 0$ and $b^2 - ac - 2td \neq 0$ are essentially distinct. It is also observed, from the form of G_6' above, that when t vanishes, $a_1 = e_3 e_1^2 a$, so that it is necessary then to distinguish further according to the vanishing or non-vanishing of a. *We are thus led to the following classification of the systems* (21) *with respect to* G_6:

$$\text{I:} \quad t = 0, \ b^2 - ac - 2td = 0, \ a = 0;$$

$$\text{II:} \quad t = 0, \ b^2 - ac - 2td = 0, \ a \neq 0;$$

$$\text{III:} \quad t = 0, \ b^2 - ac - 2td \neq 0, \ a = 0;$$

$$\text{IV:} \quad t = 0, \ b^2 - ac - 2td \neq 0, \ a \neq 0;$$

$$\text{V:} \quad t \neq 0, \ b^2 - ac - 2td = 0;$$

$$\text{VI:} \quad t \neq 0, \ b^2 - ac - 2td \neq 0.$$

No member of one of these classes can be transformed into a member of another class; while systems belonging to the same class are equivalent.

The last statement may be proved by showing that for each class it is possible to set up a canonical form to which all the members of that class are equivalent. We omit detailed discussion and merely state the *canonical equations with their solutions* : *

$$\text{I:} \qquad y'' = 2, \qquad\qquad y = x^2 + \alpha x + \beta;$$

$$\text{II:} \qquad x^2 y'' = 2xy' - 2y, \qquad y = \alpha x^2 + \beta x;$$

$$\text{III:} \qquad xy'' = y', \qquad\qquad y = \alpha x^2 + \beta;$$

$$\text{IV:} \quad (x^2 - 1)y'' = 2xy' - 2y, \qquad y = \alpha x^2 + \beta x + \alpha;$$

$$\text{V:} \qquad 2yy'' = y'^2, \qquad\qquad y = (\alpha x + \beta)^2;$$

$$\text{VI:} \qquad 2yy'' = y'^2 + 1, \qquad 4\beta y = (x - \alpha)^2 + 4\beta^2.$$

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* The discussion for the cases where $t \neq 0$ is really contained in § 2. The transformations there employed, namely (23) and (26), belong to G_6. Hence (24) and (27) may be used as the canonical equations for V and VI respectively.

By a transformation of the type G_6 any system (21) can be reduced to one of these forms.

In classes I–IV, the relation (21′) between λ, μ, ν is of the first degree, while in classes V, VI the relation is of the second degree. Furthermore, classes I–V all belong to the first type described in the final theorem of § 2, while VI is identical with the second type.

We pass now to the projective group G_5. In the formulas for G'_6, we put then $e_0 = 0$, so that a is a (relative) invariant as well as t and $b^2 - ac - 2td$. The six classes considered above are of course distinct with respect to the smaller group now being considered; but V and VI each subdivide according to the vanishing or non-vanishing of a. A detailed discussion shows that the only other subdivision takes place in I; here $d = 0$ must be separated from $d \neq 0$.

We then have the following complete classification of the equations (21) *with reference to the projective group* G_5:

I_0: this is I with $d = 0$, canonical form $y'' = 0$;

I_1: this is I with $d \neq 0$;

II, III, IV: same as in the G_6 classification;

V_1: this is V with $a = 0$;

V_2: this is V with $a \neq 0$, canonical form $(2y + x^2)y'' = y'^2 + 2xy' - 2y$;

VI_1: this is VI with $a = 0$;

VI_2: this is VI with $a \neq 0$, canonical form $(2y + x^2)y'' = y'^2 + 2xy' - 2y + 1$.

As canonical forms for I_1, II, III, IV, V_1, VI_1, we may take the forms for I, II, III, IV, V, VI given in the G_6 classification above. The case I_0 might be omitted as trivial since the system does not consist of proper parabolas but of straight lines. These enter into the discussion since straight lines, as well as vertical parabolas, satisfy the equation $y''' = 0$.

In the first five of the nine classes the relation between λ, μ, ν is linear. Hence: *Of the two parameter systems of curves* $y = \lambda x^2 + \mu x + \nu$ *defined by linear relations*

$$c\lambda - b\mu + a\nu - d = 0,$$

there are five projectively distinct cases:

$$I_0: \quad a = 0, \; b = 0, \; d = 0;$$

$$I_1: \quad a = 0, \; b = 0, \; d \neq 0;$$

$$II: \quad a \neq 0, \; b^2 - ac = 0;$$

$$III: \quad a = 0, \; b^2 - ac \neq 0;$$

$$IV: \quad a \neq 0, \; b^2 - ac \neq 0.$$

The first consists merely of straight lines, the others of proper parabolas. Systems belonging to the same class are equivalent under the projective transformations G_5.

§ 4. *Geometric determinations.*

In this section we characterize geometrically the systems (21) and the various classes discussed in the preceding section. The discussion is merely outlined, but the results are complete.

Consider first the systems where $t = 0$, i. e., the systems where the relation (21′) takes the linear form (9). Such a linear relation is obtained by taking a fixed parabola * $y = \lambda_0 x^2 + \mu_0 x + \nu_0$ and a variable parabola $y = \lambda x^2 + \mu x + \nu$; the straight line through their two points of intersection is

$$(\lambda - \lambda_0)y + (\lambda_0 \mu - \mu_0 \lambda)x + \lambda_0 \nu - \nu_0 \lambda = 0;$$

and the condition that this line passes through a fixed point (x_0, y_0) is

$$(y_0 - \mu_0 x_0 - \nu_0)\lambda + \lambda_0 x_0 \mu + \lambda_0 \nu - \lambda_0 y_0 = 0;$$

which is of the required form. Hence

Any system defined by a linear relation between λ, μ, ν *may be characterized as consisting of the* ∞^2 *parabolas intersecting a fixed parabola* π_0 *in pairs of points collinear with a fixed point* P_0.

The classes described at the close of § 3 (omitting I_0) are distinguished as follows: In cases II and IV the point P_0 is finite, while in I and III it is at infinity. In I and II the point P_0 is on the parabola π_0, while in III and IV it is not so situated.

We pass now to the systems for which $t \neq 0$, i. e., the systems for which the relation (21′) is really quadratic. To characterize these we make use of the canonical forms V and VI in § 3, and the transformations G_5 and G_6.

For V, the canonical equation represents the parabolas tangent to the axis of x. The transformation G_5 converts the axis into some straight line; and a quadratic transformation G_6 converts the axis into some parabola. Therefore

Any system defined by a quadratic relation (21′) *with vanishing discriminant consists of the* ∞^2 *parabolas tangent to either a fixed straight line (case* V_1) *or a fixed parabola (case* V_2).

For VI, the canonical equation represents the parabolas having the axis of x for directrix. If a point is taken on this axis, the pairs of tangents drawn to the members of the system constitute an involution. Applying a transformation G_5, the axis of x is converted into some line and the involution property still holds for any point of this line. The effect of a transformation G_6, on the other hand, is to replace the axis of x by a parabola, and the pencil of lines through the point on the axis by a pencil of congruent parabolas through the corresponding point. The result is:

Any system defined by a quadratic relation (21′) *with non-vanishing discriminant may be obtained as follows : Take a pencil of straight lines (case* VI_1), *or a*

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* The term parabola is used throughout in the sense of vertical parabola.

pencil of congruent parabolas (case VI$_2$), *and establish an involution therein. The* ∞^2 *parabolas touching the pairs (of straight lines or parabolas) belonging to the involution constitute the required system.*

§ 5. *The corresponding surfaces.*

As was remarked in the introduction, the fact that an equation is of the cubic form (1) does not of itself show that it represents the geodesics of a surface. But for the systems of parabolas obtained we shall find that corresponding surfaces do exist.

At the close of § 2, it was seen that with respect to point transformation the systems are of two distinct types. In the first type, the equation is equivalent to $y'' = 0$; so that, by the familiar BELTRAMI theorem, the surfaces are merely those of constant curvature.

It remains then to discuss only the second type, whose canonical form is

$$(27) \qquad\qquad 2yy'' = y'^2 + 1.$$

To find a corresponding surface, it is sufficient to note that, when the element of length of a surface is written in the isothermal form

$$(28) \qquad\qquad ds^2 = E(du^2 + dv^2),$$

the differential equation of the geodesics is

$$(29) \qquad\qquad v'' = - L_u v'^3 + L_v v'^2 - L_u v' + L_v,$$

where

$$L = \tfrac{1}{2} \log E.$$

This becomes of the same form as (27) if L is $\tfrac{1}{2} \log v$. The corresponding element is then

$$(30) \qquad\qquad ds^2 = v(du^2 + dv^2).$$

Hence if the element of length of a surface can be reduced to the form (30), and the point (u, v) of the surface be represented by the point $x = u$, $y = v$ in the plane, then the geodesics of the surface are pictured by the parabolas (27) having the axis of x for directrix.

If two surfaces S and S_1 can be represented upon a plane so that the geodesics are pictured by the same curves, it follows that it is possible to establish a point-to-point correspondence between S and S_1 so that the geodesics correspond. When this is true we may say that S and S_1 are geodesically equivalent. The problem of finding all pairs of geodesically equivalent surfaces was solved by DINI and LIE. The only cases where S_1 is not applicable on (isometric to) S, or a surface homothetic to S, arise for the LIOUVILLE surfaces, whose elements are reducible to the form

$$ds^2 = \{U(u) + V(v)\}(du^2 + dv^2).$$

The element (30), obtained above, is of this type, hence there exist essentially distinct classes of surfaces with the same geodesic representation. It happens that LIE, in illustration of the general theory due to DINI and himself, has given the discussion of a particular case equivalent to (30). Through an oversight, however, LIE overlooked one form of solution.*

The resulting surfaces coincide, as KOENIGS has shown, with those surfaces whose geodesic equation, $\Delta\theta = 1$, admits precisely three homogeneous integrals of the second degree. Another equivalent problem is the determination of surfaces applicable on surfaces of revolution and having an element reducible to the LIOUVILLE form in an infinite number of ways. We may therefore make use of the forms obtained by RAFFY in his discussion of this question.†

If a surface can be represented geodesically upon a surface with the element (30), then its element is reducible to (30), or to one of the forms

$$(31) \qquad ds^2 = \left(\frac{P}{v^2} + Q\right)(du^2 + dv^2),$$

$$(32) \qquad ds^2 = (Pe^v + Qe^{2v})(du^2 + dv^2),$$

$$(33) \qquad ds^2 = \frac{P + Q(e^v + e^{-v})}{(e^v - e^{-v})^2}(du^2 + dv^2);$$

so that (30), (31), (32), (33) together define a complete class of geodesically equivalent surfaces.

Applying this to our problem, we have the final result:

The only surfaces which can be represented point by point upon a plane so that the geodesics are pictured by parabolas $y = \lambda x^2 + \mu x + \nu$ are:

1°. *Surfaces of constant curvature. The system of parabolas is then linear (in the general sense), defined by a relation (21′) with vanishing discriminant δ.*

2°. *Surfaces of variable curvature whose linear element is reducible to one of the forms (30), (31), (32), (33). These are all geodesically equivalent; and are applicable on surfaces of revolution. The corresponding system of parabolas is essentially quadratic, defined by a relation (21′) whose discriminant does not vanish.*

There is no difficulty in finding the explicit equations of the surfaces of revolution whose elements are of the above form. We give the result for the simplest case. The class represented by (30) is applicable on any one of the surfaces of revolution

$$(34) \qquad x = h\sqrt{v}\cos\frac{u}{h}, \qquad y = h\sqrt{v}\sin\frac{u}{h}, \qquad z = \int\sqrt{\frac{4v^2 - h^2}{4v}}\,dv,$$

* KOENIGS, *Mémoire sur les lignes géodésiques*, Mémoires des Savants Etrangeres, vol. 31 (1894), p. 22.

† RAFFY, Comptes Rendus, vol. 108 (1889), p. 493; DARBOUX, *Théories des surfaces*, vol. 3 (1894), p. 39.

where h is an arbitrary constant. The equation of the meridian curve is

(34')
$$z = h^3 \int \sqrt{4x^4 - h^6}\, dx.$$

Another geometric characterization of our surfaces may be obtained by making use of WEINGARTEN's theorem: any surface applicable on a surface of revolution (exclusive of the catenoid) is a nappe of the evolute surface of a W-surface. The following simple result is obtained without difficulty. Take a surface whose principal radii are connected by the relation

$$R_1 + 2R_2 = 0,$$

and construct the nappe of its evolute corresponding to the radius R_1.* This gives surface with the element (30).†

§ 6. *Conformal representation.*

Having investigated all cases where a surface is capable of representation upon a plane so that the geodesics correspond to parabolas, we now determine those cases where the representation is conformal.

For this purpose, we recall that the most general conformal representation of any surface on a plane is obtained by putting ds^2 into the isothermal form (28) and setting $x = u$, $y = v$. The plane curves corresponding to the geodesics are then defined by an equation of the form

(35)
$$y'' = -L_x y'^3 + L_y y'^2 - L_x y' + L_y.$$

Comparing this with (1), we may state the useful

LEMMA. *In order that a system of plane curves shall correspond by conformal representation to the geodesics of a surface, it is necessary and sufficient that the equation of the system be of the cubic form*

(1)
$$y'' = A y'^3 + B y'^2 + C y' + D,$$

and that the coefficients satisfy the relations

(36)
$$A = C, \qquad B = D, \qquad A_y + B_x = 0.$$

The only equations (21) which satisfy these conditions are found to be

(37)
$$(2ty + c)y'' = t(y'^2 + 1).$$

If $t = 0$, this equation reduces to $y'' = 0$, so that the system consists of straight lines. The only surfaces whose geodesics can be represented conformally by straight lines are, it is known, the developable surfaces.

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* The other nappe has an element of the form $ds^2 = v^4(du^2 + dv^2)$.

† A more general theorem is given by LIE, Mathematische Annalen, vol. 20 (1882), p. 389.

If $t \neq 0$, the equation, by a translation of the x axis, may be reduced to (27). The corresponding surfaces are then represented by (30). Our result is therefore:

If a surface is capable of being represented conformally upon a plane in such a manner that its geodesics are pictured by parabolas $y = \lambda x^2 + \mu x + \nu$, then either (case 1°) the surface is developable, and the representing curves are straight lines; or (case 2°) the surface belongs to the class (30) of surfaces applicable on the surface of revolution (34), and the representing curves are parabolas with a common directrix.

§ 7. *Groups and representations.*

We have obtained all the systems of parabolas and all the surfaces connected with our problem. It remains now to determine all the possible representations.

Consider first the case of surfaces of constant curvature. The geodesics of such a surface, being equivalent to straight lines $y'' = 0$, admit an eight-parameter group, isomorphic to the projective group of the plane. Hence, for a given surface and a given system of parabolas belonging to the first type described at the close of § 2, there are ∞^8 possible representations. The total number of systems of this type is, however, ∞^3. Therefore:

A surface of constant curvature may be represented upon the plane so that its geodesics are pictured by parabolas (2) in ∞^{11} ways. None of these representations is conformal except when the curvature is zero; then there are ∞^4 conformal representations.

The corresponding discussion for the second type of surfaces obtained (those of variable curvature) is not so simple. We have seen that the system of parabolas may then be put into the canonical form (27). Our first step is to examine the group of point transformations which this equation admits.

If the equation (27) is invariant under an infinitesimal transformation with the symbol

$$\xi \frac{\partial t}{\partial x} + \eta \frac{\partial t}{\partial y},$$

the general theory of groups gives the condition *

$$\{\eta + y(\eta_y - 2\xi_x - 3\xi_y y')\}(1 + y'^2) - 2yy'\{\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2\}$$
$$+ 2y^2\{-\xi_{yy}y'^3 + (\eta_{yy} - 2\xi_{xq})y'^2 + (2\eta_{xy} - \xi_{xx})y' + \eta_{xx}\} = 0.$$

Equating the coefficients of the various powers of y' to zero, we find the following system of equations for the determination of the unknown functions ξ, η:

$$2y\xi_{yy} + \xi_y = 0, \qquad \eta - y\eta_y + 2y^2\eta_{yy} - 4y^2\xi_{xy} = 0,$$
$$3\xi_y + 2\eta_x + 2y(\xi_{xx} - 2\eta_{xy}) = 0, \qquad \eta + y(\eta_y - 2\xi_x) + 2y^2\eta_{xx} = 0.$$

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* LIE-SCHEFFERS, *Differentialgleichungen*, p. 363.

We shall omit the somewhat long but not particularly difficult discussion of these equations and simply state the solution

$$(38) \qquad \xi = a_1 x^2 + a_2 x + a_3, \qquad \eta = 2a_1 xy + a_2 y.$$

Hence the equation (24) admits three independent infinitesimal transformations with the symbols

$$(39) \qquad \frac{\partial f}{\partial x}, \qquad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}, \qquad x^2 \frac{\partial f}{\partial x} + 2xy \frac{\partial f}{\partial y}.$$

The finite equations of the continuous group generated by these infinitesimal transformations may be found by the integration of the simultaneous system

$$\frac{dx_1}{a_1 x_1^2 + a_2 x_1 + a_3} = \frac{dy_1}{2a_1 xy + a_2 y} = d\tau,$$

with the initial conditions $\tau = 0$, $x_1 = x$, $y_1 = y$.

When a_1 vanishes, it is found that

$$(40) \qquad x_1 = a'x + b', \qquad y_1 = a'y.$$

When a_1 does not vanish, it may be taken equal to unity, and the integrals, after putting $\tau = \log t$, take the form

$$(41) \qquad \frac{x_1 + r}{x_1 + s} = t \frac{x + r}{x + s}, \qquad \frac{(x_1 + r)(x_1 + s)}{y_1} = \frac{(x + r)(x + s)}{y}$$

or, explicitly,

$$(41') \quad x_1 = \frac{(st - r)x + rs(t - 1)}{(1 - t)x + s - rt}, \qquad y_1 = \frac{t(s - r)^2 y}{\{(1 - t)x + s - rt\}^2}.$$

Hence, equation (27), or the system of parabolas with a common directrix, admits the three-parameter continuous groups H_3 represented by (40) and (41). The transformations (40) constitute a two-parameter conformal subgroup.

Consider now the possible representations of a surface S defined by

$$ds^2 = v(du^2 + dv^2).$$

We have already seen that, by the representation

$R:$ $\qquad\qquad\qquad x = u, \qquad y = v,$

the geodesics are pictured by the system (27),

$\sigma:$ $\qquad\qquad\qquad 2yy'' = y'^2 + 1.$

Since σ is transformed into itself by the group H_3, it follows that if R is combined with one of these plane transformations, the resulting representation will still picture the geodesics of S by the system σ. We thus obtain ∞^3 repre-

sentations RH_3, corresponding to the one system σ. Of these, a double infinity, namely RG_2, are conformal.

But there are ∞^4 systems of the second type described in § 2, any one of which may be used instead of the system σ. All these are equivalent to σ with respect to the group G_6 of § 3. To obtain all the systems from σ it is in fact sufficient to employ the four-parameter subgroup

$$G_4: \qquad x_1 = x, \qquad y_1 = e_0 x^2 + e_3 y + e_4 x + e_5.$$

The most general representation of S is therefore obtained by combining G_4 with the representations already found. The result may be indicated by RH_3G_4, and involves seven arbitrary constants. This yields no new conformal representations since G_4 contains no conformal transformations except identity.

If a surface belongs to the class defined by (30) *it may be represented upon the plane so that its geodesics are pictured by parabolas in ∞^7 ways, of which ∞^2 are conformal. The general representation is of the form RH_3G_4, while the conformal is RG_2.*

Let us pass now to the other surfaces belonging to the second type of § 5. Such a surface is geodesically equivalent to those just discussed. The geodesics will still admit a three-parameter group, so that there are ∞^3 representations for the one system σ. From § 6, none of these will be conformal. Our result is:

If a surface belongs to the classes defined by (31), (32), *or* (33), *it is possible to represent it in ∞^7 ways upon a plane so that its geodesics are pictured by parabolas. None of these representations is conformal.*

§ 8. *Additional results.*

We may extend the results obtained by making use of the following

Lemma. If an equation of cubic type (1) is an integral of an equation of the form

$$(42) \qquad y^{(n)} + P_1 y^{(n-1)} + \cdots + P_{n-1} y' + P_n = 0,$$

where the coefficients are functions of x, y, then $A = 0$, that is, (1) reduces to a quadratic.

To prove this, it is sufficient to observe that, when the values of y'', y''', $\cdots$, $y^{(n)}$ obtained from (1) in terms of y' are substituted in (42), the coefficient of the highest power of y' is a numerical multiple of A^{n-1},

In particular, if the cubic equation is of the form (35), arising in connection with conformal representation, it follows that $L_x = 0$, so that L is a function of y alone. The corresponding linear element (28) then belongs to surfaces applicable on surfaces of revolution.

If a surface can be represented conformally on a plane so that the curves depicting its geodesics satisfy an equation of the form (42) *(of any order but linear in the derivatives), then its element is necessarily of the form*

$$(43) \qquad\qquad ds^2 = f(v)(du^2 + dv^2),$$

so that the surface is applicable on a surface of revolution.

The equation $y''' = 0$, which has been discussed, is of the form (42). We have seen (§ 6) that the corresponding surfaces are the developables and those with the element (30). We shall now show that no new surfaces arise in connection with any equation (42) of the third order.

The equation of the geodesics of (43) is

$$(44) \qquad\qquad y'' = (1 + y'^2)Y,$$

where Y is a function of y, namely,

$$(44') \qquad\qquad Y = \frac{f'(y)}{2f(y)}.$$

By successive differentiation, we find

$$y^{\mathrm{III}} = y'(1 + y'^2)Y_1, \qquad\qquad \text{where } Y_1 = 2Y^2 + Y';$$

$$(45) \quad y^{\mathrm{IV}} = (1 + y'^2)(Y_2 y'^2 + Y_3), \quad \text{where } Y_2 = 3YY_1 + Y_1', \quad Y_3 = YY_1;$$

$$y^{\mathrm{V}} = y'(1 + y'^2)(Y_4 y'^2 + Y_5), \text{ where } Y_4 = 4YY_2 + Y_2',$$
$$Y_5 = 2Y(Y_2 + Y_3) + Y_3'; \text{ etc.}$$

Taking the equation (42) for $n = 3$, and substituting the values of y' and y''', we have

$$y'(1 + y'^2)Y_1 + (1 + y'^2)YP_1 + y'P_2 + P_3 = 0.$$

Equating the coefficients of powers of y', we obtain a simple set of conditions, with the solution

$$P_1 = 0, \qquad P_2 = 0, \qquad P_3 = 0, \qquad Y_1 = 0.$$

The first three values show that (42) reduces to $y''' = 0$; and the last shows that Y is either a constant or of the form $(2y + c)^{-1}$. This leads, of course, to the surfaces considered in § 6.

The only case in which an equation of the third order of the form (42) can be satisfied by a system of curves corresponding to the geodesics of a surface in conformal representation, is the case $y''' = 0$; the surfaces are then the developables and the class (30) described in § 6.

The examination of the equations $y^{\mathrm{IV}} = 0$, $y^{\mathrm{V}} = 0$, $y^{\mathrm{VI}} = 0$ leads to the same conclusion: no new surfaces are obtained. The same is *probably* true of all equations of the form $y^{(n)} = 0$. There are, however, equations of the fourth order of the form (42) which lead to new solutions. We shall discuss these on another occasion.

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